6.3 Vectors in the Plane

Many quantities in geometry and physics, such as area, time, and temperature, can be represented by a single real number.

Other quantities, such as force and velocity, involve both magnitude and direction and cannot be completely characterized by a single real number.

To represent such a quantity, we use a directed line segment. The directed line segment $\overrightarrow{PQ}$ has initial point $P$ and terminal point $Q$ and we denote its magnitude (length) by $\|PQ\|$. 

![Diagram: Directed line segment with initial point P and terminal point Q]
Vector Representation by Directed Line Segments

Let \( \mathbf{u} \) be represented by the directed line segment from \( P = (0,0) \) to \( Q = (3,2) \), and let \( \mathbf{v} \) be represented by the directed line segment from \( R = (1,2) \) to \( S = (4,4) \). Show that \( \mathbf{u} = \mathbf{v} \).

Using the distance formula, show that \( \mathbf{u} \) and \( \mathbf{v} \) have the same length. Show that their slopes are equal.

\[
\|\mathbf{u}\| = \sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13}
\]

\[
\|\mathbf{v}\| = \sqrt{(4-1)^2 + (4-2)^2} = \sqrt{13}
\]

Slopes of \( \mathbf{u} \) and \( \mathbf{v} \) are both \( \frac{2}{3} \).
Component Form of a Vector

The component form of the vector with initial point \( P = (p_1, p_2) \) and terminal point \( Q = (q_1, q_2) \) is

\[
\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = v
\]

The magnitude (or length) of \( v \) is given by

\[
\|v\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}
\]
Find the component form and length of the vector $v$ that has initial point $(4, -7)$ and terminal point $(-1, 5)$

Let $P = (4, -7) = (p_1, p_2)$ and $Q = (-1, 5) = (q_1, q_2)$.

Then, the components of $v = \langle v_1, v_2 \rangle$ are given by

$v_1 = q_1 - p_1 = -1 - 4 = -5$
$v_2 = q_2 - p_2 = 5 - (-7) = 12$

Thus, $v = \langle -5, 12 \rangle$

and the length of $v$ is

$$\|v\| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13$$
Vector Operations

The two basic operations are **scalar multiplication** and **vector addition**. Geometrically, the product of a vector $v$ and a scalar $k$ is the vector that is $|k|$ times as long as $v$. If $k$ is positive, then $kv$ has the same direction as $v$, and if $k$ is negative, then $kv$ has the opposite direction of $v$. 

![Diagram](image-url)
Definition of Vector Addition & Scalar Multiplication

Let $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$ be vectors and let $k$ be a scalar (real number). Then the sum of $u$ and $v$ is

$$u + v = \langle u_1 + v_1, u_2 + v_2 \rangle$$

and scalar multiplication of $k$ times $u$ is the vector

$$ku = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle$$
Vector Operations

Ex. Let $v = \langle -2, 5 \rangle$ and $w = \langle 3, 4 \rangle$. Find the following vectors.

a. $2v$

$$2v = \langle -4, 10 \rangle$$

b. $w - v$

$$w - v = \langle 3 - (-2), 4 - 5 \rangle = \langle 5, -1 \rangle$$
Writing a Linear Combination of Unit Vectors

Let \( u \) be the vector with initial point \((2, -5)\) and terminal point \((-1, 3)\). Write \( u \) as a linear combination of the standard unit vectors of \( \mathbf{i} \) and \( \mathbf{j} \).

**Solution**

\[
\begin{align*}
   u &= \langle -1 - 2, 3 + 5 \rangle \\
      &= \langle -3, 8 \rangle \\
      &= -3\mathbf{i} + 8\mathbf{j}
\end{align*}
\]

Graphically, it looks like…
Writing a Linear Combination of Unit Vectors

Let \( u \) be the vector with initial point \((2, -5)\) and terminal point \((-1, 3)\). Write \( u \) as a linear combination of the standard unit vectors \( i \) and \( j \).

Begin by writing the component form of the vector \( u \).

\[
\begin{align*}
\mathbf{u} &= \langle -1 - 2, 3 - (-5) \rangle \\
\mathbf{u} &= \langle -3, 8 \rangle \\
\mathbf{u} &= -3i + 8j
\end{align*}
\]
Unit Vectors

\( u = \text{unit vector} = \frac{v}{\|v\|} = \left( \frac{1}{\|v\|} \right)^v \)

Find a unit vector in the direction of \( v = \langle -2, 5 \rangle \)

\[
\frac{v}{\|v\|} = \frac{\langle -2, 5 \rangle}{\sqrt{(-2)^2 + (5)^2}} = \frac{1}{\sqrt{29}} \langle -2, 5 \rangle = \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle
\]
Let $u = -3i + 8j$ and let $v = 2i - j$. Find $2u - 3v$.

$$2u - 3v = 2(-3i + 8j) - 3(2i - j)$$

$$= -6i + 16j - 6i + 3j$$

$$= -12i + 19j$$