3.7
Optimization Problems
Ex. 1  A manufacturer wants to design an open box having a square base and a surface area of 108 in\(^2\). What dimensions will produce a box with maximum volume?

Since the box has a square base, its volume is

\[ V = x^2h \]

Note: We call this the primary equation because it gives a formula for the quantity we wish to optimize.

The surface area = the area of the base + the area of the 4 sides.

\[ \text{S.A.} = x^2 + 4xh = 108 \]

We want to maximize the volume, so express it as a function of just one variable. To do this, solve

\[ x^2 + 4xh = 108 \text{ for } h. \]
\[ h = \frac{108 - x^2}{4x} \]

Substitute this into the Volume equation.

\[ V = x^2 h = x^2 \left( \frac{108 - x^2}{4x} \right) = 27x - \frac{x^3}{4} \]

To maximize \( V \) we find the derivative and it’s C.N.’s.

\[ \frac{dV}{dx} = 27 - \frac{3x^2}{4} = 0 \quad 3x^2 = 108 \quad C.N.'s \quad x = \pm 6 \]

We can conclude that \( V \) is a maximum when \( x = 6 \) and the dimensions of the box are 6 in. x 6 in. x 3 in.
Procedures for solving Applied Minimum and Maximum Problems

1. Assign symbols to all given quantities and quantities to be determined.
2. Write a primary equation for the quantity to be maximized or minimized.
3. Reduce the primary equation to one having a single independent variable. This may involve the use of secondary equation.
4. Determine the domain. Make sure it makes sense.
5. Determine the max or min by differentiation.
Find the points on the graph of \( y = 4 - x^2 \) that are closest to the point \((0,2)\).

What is the distance from the point \((x,y)\) and \((0,2)\)?

\[
d = \sqrt{(x-0)^2 + (y-2)^2}
\]

We want this dist. to be minimized.

\[
d' = \frac{4x^3 - 6x}{2(x^4 - 3x^2 + 4)^{\frac{3}{2}}}
\]

\[
= \sqrt{x^2 + (4 - x^2 - 2)^2}
\]

\[
= \sqrt{x^4 - 3x^2 + 4}
\]
\[ d' = \frac{4x^3 - 6x}{2(x^4 - 3x^2 + 4)^{\frac{1}{2}}} = 2x(2x^2 - 3) = 0 \]

C.N.'s \[ 0, \pm \frac{\sqrt{6}}{2} \]

imaginary

\[ \begin{array}{cccc}
- & + & - & + \\
\text{dec. } & -\frac{\sqrt{6}}{2} & \text{inc. } & 0 & \text{dec. } & \frac{\sqrt{6}}{2} & \text{inc. }
\end{array} \]

1st der. test

\[ \left( -\frac{\sqrt{6}}{2}, \frac{5}{2} \right) \]

minimum

\[ \left( \frac{\sqrt{6}}{2}, \frac{5}{2} \right) \]

minimum

Two closest points.